Notes on ME525 Applied Acoustics Lecture 7 Winter 2024

Complex Intensity, Active and Reactive Intensity

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Complex Intensity, Active and Reactive Intensity

Returning to the red and blue "envelope" curves (Fig. 1) first shown for the Jacobsen data in Lecture 6 Recall that at $kr << 1$ the situation is characterized by reactive intensity, and at $kr >> 1$ the situation is characterized by active intensity. These curves emerge through the concept or *complex intensity* $\vec{I}_c = \frac{1}{2} p \vec{u}^\star$ which was first formulated by Heyser (1986), and is discussed further in Fahy (1995) and Jacobsen and Juhl (2015).

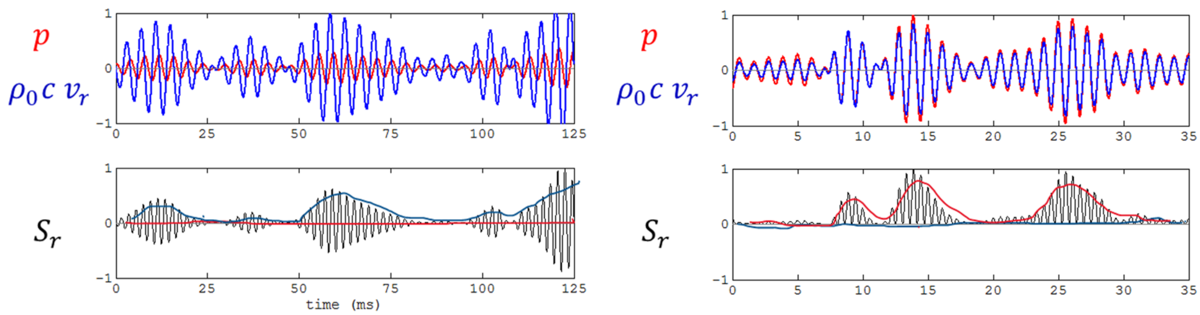

Figure 1: Jacobsen Ezperiment left: near field with $kr << 1$, right: far field with $kr >> 1$. (See also Lecture 6)

This is subtle concept, best first demonstrated by a model. Use our standard model for the acoustic pressure from a spherical wave $p(r,t) = \frac{A}{r} e^{ikr - i\omega t}$, and form $\frac{1}{2} p \vec{u}^\star$, using in this case only a radial component u_r for $\vec{u}$. This yields

$$I_c = \frac{|A|^2}{2\rho_0 c} \left(\frac{1}{r^2} - i \frac{1}{r^2 kr} \right). \tag{1}$$

so in this case, complex intensity I_c has only one component in the radial direction to mirror u_r. Confirm Eq.(1) yourself.

The real part I_c equals the *active intensity* and identify $I = \mathrm{Re}\{I_c\}$. The imaginary part equals the *reactive intensity* and identify $Q = \mathrm{Im}\{I_c\}$ (in the general case these are vectors $\vec{I}$ and $\vec{Q}$). Thus according to this model active intensity (limited to the r direction) is

$$I_r = \frac{|A|^2}{2\rho_0 c r^2} \tag{2}$$

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Recall the corresponding Umov vector for this same model for pressure (Lecture 6)

$$S_r(r,t) = \frac{|A|^2}{r^2 \rho_0 c}\{\cos^2(kr - \omega t + \phi_A) - \frac{\cos(kr - \omega t + \phi_A)\sin(kr - \omega t + \phi_A)}{kr}\} \tag{3}$$

Observe: the time average of $S_r(r,t)$ identified formally as $\langle S_r(r,t)\rangle$ also yields the result for I in Eq.(2). Since the problem involves harmonic variables of single frequency f where $\omega = 2\pi f$, time average is carried out over a period $T = 1/f$. Check out the time average: the first time equals 1/2 and the second terms = 0. Henceforth associate *active intensity* $\vec{I}$ as a time average, in some reasonable sense, of the Umov vector.

With this simple model the result $\langle S_r(r,t)\rangle$ no longer shows time variation, only the spatial variation by way of range r. With real data there can be slowly-varying changes, for example, going back to the Jacobsen data, case $kr >> 1$, observe the red line (a rough sketch we added to the data) is describing in some sense a kind of "running average" of the Umov vector. For example, time variation could reflect changes in active intensity caused by changing the volume of the speaker used in the experiment.

The imaginary part, *reactive intensity*, is more subtle. The second term of $S_r(r,t)$ in Eq. (3)–the "0 time-averaged part" –is suggestive of reactive intensity insofar as this term has a different range dependence, going as $\sim 1/r^3$, compared to active intensity that goes as $\sim 1/r^2$. But using complex intensity we find it exactly as

$$Q_r = -\frac{|A|^2}{2\rho_0 c r^2 kr} \tag{4}$$

(Note this subtle point: the sign of reactive intensity is not of physical significance, and depends on which convention $e^{\pm i\omega t}$ is used.)

How does this running average idea work with Q_r? Interpret reactive intensity as some measure of the "strength" of the Umov vector, even though that vector may have *time average* of zero. In other words, interpret reactive intensity as the strength of an Umov vector that oscillates in sign. For example, in Fig. 1, case $kr << 1$ observe the blue line tracing the envelope of the Umov vector, which is seen upon inspection to have a near-zero time average. In constrast, for Fig. 1, case $kr >> 1$ the Umov vector clearly has a non-zero time average, is basically of one sign, and thuse the reactive intensity is small.

In Fig 1 the active (red) and reactive (blue) lines are sketched in (more or less guessed at, since they are not given in original article). Another better demonstratio is one based on our own measurements at the Army Research Laboratory's anachoic chamber made by my colleague Dr. David Dall'Osto, and me. Besides testing the instrument we were developing at the time, another strong motivation was for us to duplicate the Jacobsen experiment. Shown are results from two ranges, 0.28 m and 2.28 m, from a speaker source transmitting at frequency 160 Hz, with a typical conditions for air of c = 343 m/s and ρ_0 = 1.2 kg/m^3.

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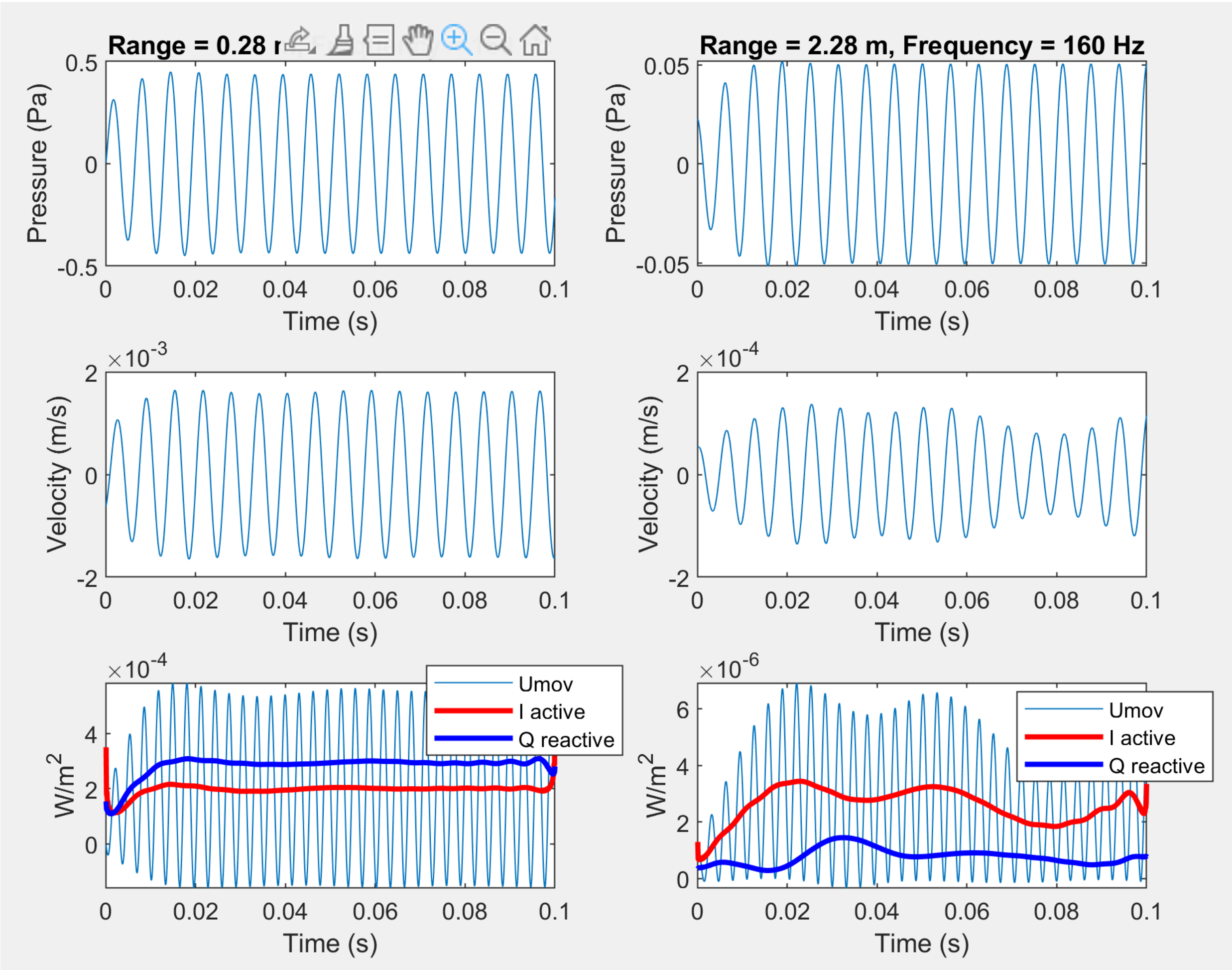

Figure 2: Acoustic pressure (top) acoustic velocity (middle) and intensity measures (bottom) based on measurements at 160 Hz, at two ranges completed at the Army Research Laboratory anachoic chamber, Dall'Osto and Dahl

At range 0.28 m, $kr \sim 0.8$, not quite $kr << 1$, but clearly not $kr >> 1$. We anticipate a mixture of active and reactive intensity. This is suggested by inspecting the pressure and velocity time series, for which there is a small difference phase between pressure and velocity. Taking a pure average $\langle S_r(r,t) \rangle$ over this 0.1 s time period, yields one value = 1.96 10^{-4} W/m^2, and the active intensity I_r (red line), approximately captures this number, though varies somewhat at the start. The active intensity is again single component in the r-direction. The reactive intensity Q_r (blue line) is mixed in with the active component and evidently higher strength or value. At range 2.28 m $kr \sim 6$, notice that oscillations in pressure and velocity align much better, though not perfectly, and we can anticipate the observation that Q_r will have diminished considerably relative to I.

How do we find the complex intensity, real (red line) and imaginary (blue line) parts when working with this type of real-valued measurement data? In matlab a simple solution is to form the *Hilbert transform pair* of the data using, `v_complex = hilbert(v);` where `v` is the matlab variable representing a time series of velocity, and `v_complex` is the Hilbert transformed pair

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result. Recover a conjugate form of the velocity in matlab using `conj(v_complex)`, or recover the original real-valued time series using `real(v_complex)`.

Another example from real data involves measurement of a sonar pulse at range 500 m (Fig 3). In this case the Umov vector is in 3-components x, y, z. The S_x and S_y vectors point in opposite directions–telling us something about the bearing of this pulse relative to the sensor. Notice how the vertical component S_z is ocscillatory, indicative of reactive intensity. This is typical in underwater acoustics and we'll continue exploring this data a bit more next week.

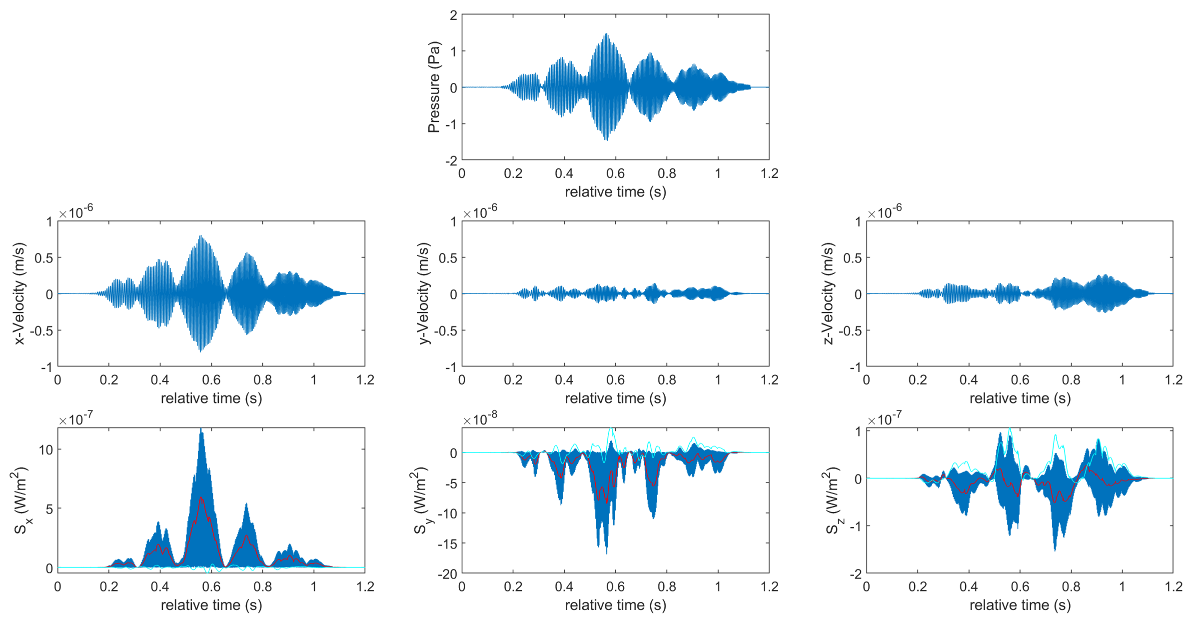

Figure 3: A frequency-modlulated (FM) pulse from a sonar measured at range 500 m from a vector sensor. Top row: Acoustic pressure, Middle row: aoustic velocity in x, y, z directions, bottom row: Umov vector $S_{x,y,z}$ and corresponding active (red) and reactive (cyan) intensities.

Summarizing:

- When acoustic pressure and velocity are 90° out of phase, as in the Jacobsen data for $kr << 1$ there exists reactive intensity, $\langle S_r(r,t)\rangle \sim 0$, and reactive intensity Q will describe the envelope of $S_r(r,t)$

- When acoustic pressure and velocity in phase, as in the Jacobsen data for $kr >> 1$ there exists active intensity I, $\langle S_r(r,t)\rangle$ is non-zero,

the above given in term of a single radial component, but in general there is $\vec{S}$, $\vec{I}$ and $\vec{Q}$.

We have now encountered multiple definitions relating to word intensity, all of which should have as their basic dimension Watts/m^2, or J/sec/m^2. Intensity is in general a vector quantity for which the following forms have been introduced

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- Umov vector $\vec{S}$
$$\vec{S}(r,t) = Re\{p(r,t)\}Re\{\vec{u}(r,t)\} \tag{5}$$

- Complex intensity $\vec{I}_c$
$$\vec{I}_c = \frac{1}{2}p(r,t)\vec{u}^{\star}(r,t) \tag{6}$$

- *active* intensity $\vec{I} = \mathrm{Re}\{\vec{I}_c\}$ and *reactive* intensity $\vec{Q} = \mathrm{Im}\{\vec{I}_c\}$

- *plane wave* intensity $\frac{p_{rms}^2}{\rho_0 c}$ This is sometimes referred to as plane wave intensity as it is precisely the intensity one finds from a plane wave. The expression is handy to use with real data–but be careful with it usage (let's examine in a small homework problem)

References

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ME525 Applied Acoustics Lecture 8, Winter 2024 Concluding remarks on Active and Reactive Intensity Radiated acoustic power from spherical source, The $ka << 1$ limit, point source and Green's function

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Concluding remarks on Active and Reactive Intensity

The following are some concluding remarks on active $\vec{I}$ and reactive $\vec{Q}$ intensity; these concepts take practice to sink in (especially for $\vec{Q}$). We'll not have time to study them further in ME 525, so the take home message at this point is: when pressure and velocity are in phase, there exists active intensity. When they are out of phase, there exists active intensity, and there can be many gradations of this.

Back to the remarks: The acoustic vector field $\vec{v}$ should now be one of the familiar acoustic small variables, e.g., as used in acoustic radiation from a spherical source, where $\vec{v} = v_r$. In acoustics we understand that $\nabla \times \vec{v} = 0$, in other words the curl of the acoustic velocity field equals zero and acosutic velocity field is said to be *irrotational*. But this is not the case for active intensity $\vec{I}$ equal to Umov vector time average, $\langle \vec{S} \rangle$ or derived from complex intensity, $\vec{I}_c = \frac{1}{2} p \vec{u}^\star$, then taking the real part $\vec{I} = \text{Re}\{\vec{I}_c\}$.

An example (Fig. 1) comes from our research in underwater sound. Here an underwater sound source is towed at speed 1.6 m/s transmitting at frequency 43 Hz. The source is moving toward, or closing, on the vector acoustic receiver, so there is slight up (Doppler) shift in frequency to $\sim$ 43.04 Hz. The received acoustic field, highly filtered to be close to 43.04 Hz, exemplifies *narrow band* sound, e.g., sound that can analyzed assuming the time dependence is $e^{-i\omega t}$ where $\omega = 2\pi 43.04$. We thus exploit $\vec{I}_c$ and study $\vec{I}$ and its companion $\vec{Q}$.

At a point in space and at a particular frequency, sound can undergo destructive interference (discussed in future lectures), with the sound field almost vanishing or fading. For example, inspect closely the FM pulse from Lecture 7 where the frequency increases in time from about 150 to 250 Hz. The pressure envelope modulates with time as some frequencies fade in and out. At such points of strong fading an active intensity vortex can form, and an example is depicted in Fig. 1 (a) which is generated with model based on the conditions of experiment. Streamlines trace out the course of the changing $\vec{I}$ (black arrows) and show the direction of active intensity vector as it swirls around the vortex point (white disk). Clearly, $\nabla \times \vec{I}$ is not zero. To get the intensity on back on track and on its way, the vortex point has a companion stagnation point (cross mark). This combination of points is called vortex region, and field observations of this region and similar points as function of towing range, are discussed in the reference. These observations involve different combinations of measured $\vec{I}$ and $\vec{Q}$.

For comparison Fig 1 (b) and (c) show a vortex region based on modeling in air at much higher frequency (2070 Hz). Note the cm-scales compared to m-scales in Fig. 1 (a), but otherwise similar features are shown. In Fig. 1 (b) arrows corresponding to $\vec{I}$ would form (if plotted) streamlines for which the vortex and stagnation points are easily visualized. In Fig. 1 (c) streamlines of reactive intensity $\vec{Q}$ are shown; note that $\nabla \times \vec{Q} = 0$ and this vector points towards the vortex singularity.

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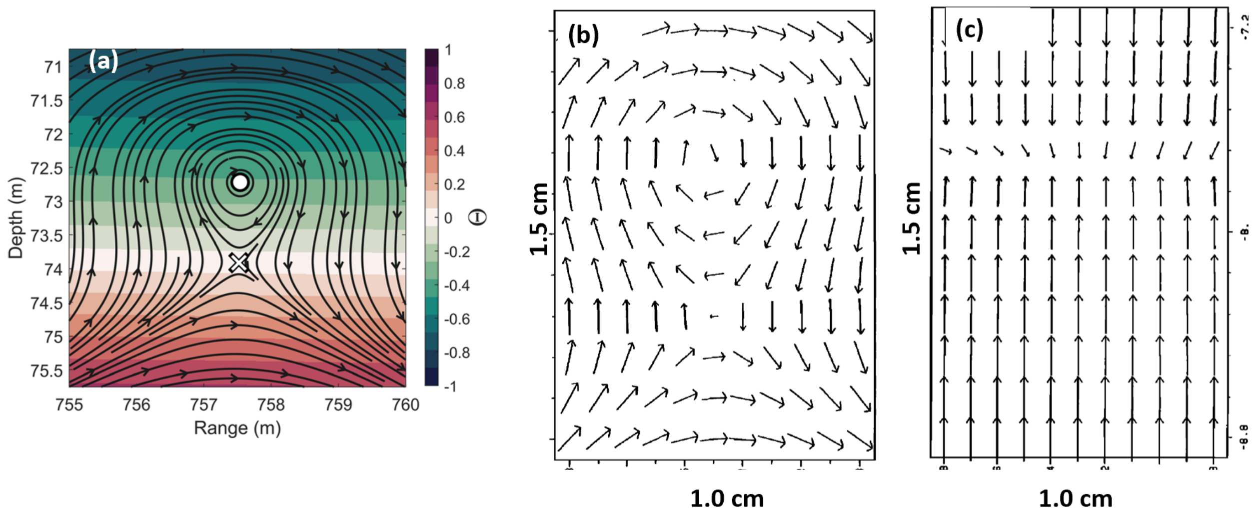

Figure 1: (a) The vortex region at range 755-760 m, and depth 71-75.5 m, from a narrow band acoustic field frequency $\sim$ 43 Hz, in waters depth 77 m. Streamlines of acoustic vector field $\vec{I}$ (energy flux streamlines) shown flowing around the vortex center (white disk) and towards the stagnation point (cross mark). Color background depicts the degree of $\nabla \times \vec{I}$. Figure from Dahl,Dall'Osto,Hodgkss, *J. Acoust. Soc. Am.*, 2023. (b) streamlines of $\vec{I}$ and (c) $\vec{Q}$ for vortex region modeled at different scale for sound in air. Figure from Mann, Tichy and Romano, *J. Acoust. Soc. Am.*, 1987.

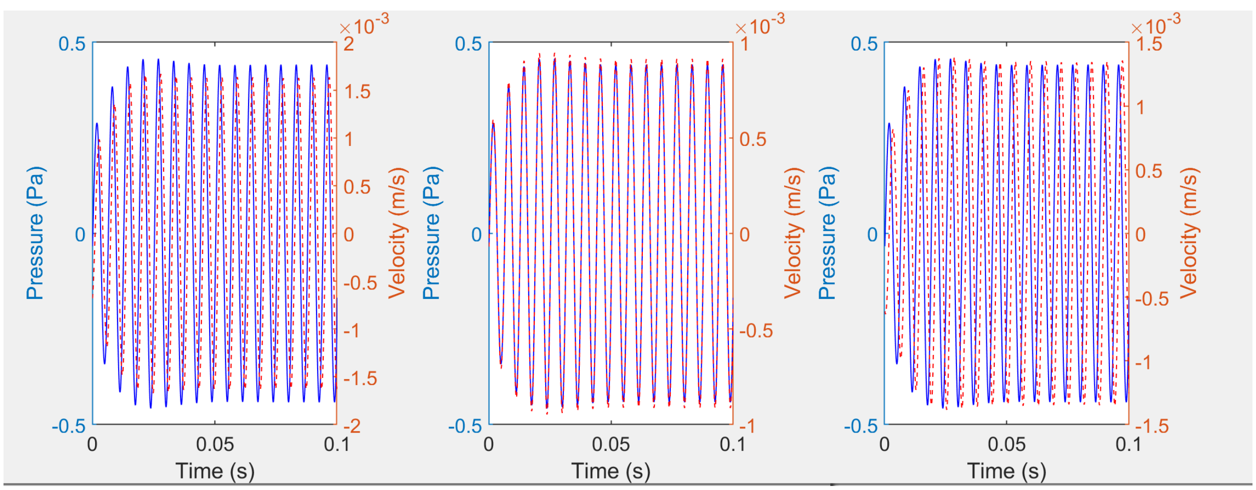

Figure 2: Further analysis of data measured at the Army Research Laboratory (Fig. 2 of Lecture 7) at $kr \sim 0.8$ based on splitting the total velocity v into in-phase component v_p and out-of-phase component v_q. Left : comparison of pressure and total velocity v . Middle: pressure and in-phase velocity (v_p), Right: pressure and out-of-phase velocity (v_q)

Let's conclude with a more practical demonstration on the take home message about pressure and velocity. Return to the data taken in Army Research Laboratory's anechoic room (Fig. 2 of Lecture 7), at range such that $kr \sim 0.8$. We split the acoustic velocity into two parts: v_p which represent a velocity that is exactly in-phase with pressure p, and v_q which represents a velocity that

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is exactly 90° out of phase with pressure, using

$$v_p = \frac{\langle pv \rangle p}{\langle p^2 \rangle}. \tag{1}$$

This amazingly simple algorithm is discussed in Stanzial *et al.* 2012. where p and v are the pressure and total velocity as shown in Lecture 7, Fig. 2 for the case $kr \sim 0.8$. The component of pressure out of phase is v_q, is defined such that $v_q = v - v_p$.

The partition of velocity into these two components is shown in Fig. 2 (this lecture). The time average of the Umov vector $\langle pv \rangle$ would be the same as taking $\langle pv_p \rangle$ as the pv_q contribution would not yield any time average. Active intensity is to be associated with pv_p, pressure and velocity clearly having the same phase relation, and reactive intensity is to associated with pv_q. The concludes our discussion on active $\vec{I}$ and reactive $\vec{Q}$ intensity.

The spherically symmetric source in $ka << 1$ limit, and the monopole source

Recall $p(r,t) = \frac{A}{r} e^{ikr - i\omega t}$ for the complex representation of a spherically symmetric pressure wave, and find the RMS pressure

$$p_{rms} = \frac{1}{\sqrt{2}} \frac{|A|}{r} \tag{2}$$

and the time-average of the Umov vector $\langle S \rangle$ for this case is

$$\langle S \rangle = \frac{p_{rms}^2}{\rho_0 c} \tag{3}$$

There is an easy way (think active intensity) and hard way to confirm Eq.(3) and means to Eq. (2) should come from inspection. This is also a measure commonly estimated with real data, and it applies generally to harmonic waves (e.g., a single frequency), but also is sometimes applied to transient sounds as in explosive waveform (multiple frequency content), and ambient noise. The intensity metric $\frac{p_{rms}^2}{\rho_0 c}$ is referred to as *plane wave* intensity because it is formally the intensity from a plane wave. One must apply some caution because measuring the pressure and forming p_{rms}^2 and dividing by $\rho_0 c$ does not generally create the vector quantity required, for example, which can be used to find total radiated acoustic power, Π from an acoustic source.

To obtain this measure in a true sense the average rate at which energy flows through a closed spherical surface of radius r that surrounds the source, a control surface S_c, is computed as follows

$$\Pi = \int_{S_c} \langle S \rangle \cdot d\vec{s} \tag{4}$$

So in general the dot product of the component of $\langle S \rangle$ normal to the differential area $d\vec{s}$ is computed and summed or integrated over the surface S_c. However using the spherically symmetric

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pressure field (pretty good model in many cases) the integral is done by inspection, yielding

$$\Pi = 2\pi \frac{|A|^2}{\rho_0 c} \tag{5}$$

Observe that the average rate of energy flow through any control surface surrounding the spherical source, or the acoustic power Π, is independent of the radius of that control surface, which is consistent with conservation of energy in a lossless medium. In practice, sound absorption can reduce the total Π (Kinsler, *et al.* 1982.)

Apply the result found earlier for a spherical source of radius a, wavenumber k, $\rho_0 c$ and radial velocity amplitude u_0 at the surface of the sphere giving complex amplitude A and find

$$\Pi = 2\pi a^2 |u_0|^2 \rho_0 c \frac{(ka)^2}{1+(ka)^2}. \tag{6}$$

It should be more obvious now that effective radiation of acoustic power for a small source as characterized by $ka << 1$ is more difficult (think of combination of small earpod and low frequency versus high frequency sounds) as the small ka limit shows that $\Pi \sim (ka)^2$.

Continuing with this spherical wave of the form

$$p(r,t) = \frac{A}{r} e^{ikr} e^{-i\omega t} \tag{7}$$

and again with boundary condition $u_r(r=a) = u_0 e^{-i\omega t}$, with A given by

$$A = \rho_0 c \; u_0 a \left(\frac{ka}{ka+i}\right) e^{-ika}, \tag{8}$$

and study the factor in parenthesis in the limit of $ka << 1$. One finds $A \approx -k\rho_0 c u_0 a^2$, based on $\frac{ka}{ka+i} = -ika$ plus order $(ka)^2$. With minor rearrangement the pressure can now be expressed as

$$p(r,t) = -i\omega(\rho_0 u_0 4\pi a^2) \frac{e^{ikr}}{4\pi r} e^{-i\omega t} \tag{9}$$

Note the $-i\omega$ (time derivative) and the $\rho_0 u_0 4\pi a^2$ (a mass) corresponding to a mass flow of (dimension M/T). Thus the strength of this acoustic source is defined by the time derivative of *mass flow*, or described another way, it is the *rate of change of mass flow* introduced per unit volume.

Next bundle everything by putting $q = -i\omega(\rho_0 u_0 4\pi a^2)$ and call this an *effective source strength*. Thus

$$p(r,t) = \frac{q}{4\pi r} e^{ikr - i\omega t} \tag{10}$$

where the source is at the center of the coordinate system and pressure is function only of radial coordinate r.

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A further idealization is made as follows: consider the hypothetical case of a becoming progressively smaller while u_0 becomes larger such that q remains constant. This is the concept of a point source or *acoustic monopole* (Pierce, 1989), for which the source is idealized to originate from a single point. The idealization is required to confine the source within an infinitely small space, or single point, however in practice any small source with time-varying mass of fluid in any small volume enclosing the source has all the attributes of a point source (Pierce, 1989).

Before moving on recall the wave equation, e.g., Eq.(10) from Lecture 3, which (in the context of that lecture) was written as

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})p = 0 \tag{11}$$

(where p is now used to represent acoustic pressure). Moving forward, the discussion with respect to modeling will most often involve harmonic variables with time dependence $e^{-i\omega t}$, and thus the operator $\frac{\partial^2}{\partial t^2}$ translates to $-\omega^2$. Based on this fact the Helmholtz form for the wave equation will be convenient, which is

$$(\nabla^2 + k^2)p = 0. \tag{12}$$

As written, Eq.(12) is homogeneous, being equal to 0, and there is no term on the right-hand-side (RHS) to represent the source. This issue is resolved in the following discussion.

The Green's function

We further generalize things to find the pressure at a *field point* $\vec{r}$, given a source at an arbitrary *source point* $\vec{r_0}$ that need not be at origin (Fig. 3) as follows:

$$p(r,t) = \frac{q}{4\pi|\vec{r} - \vec{r_0}|}e^{ik|\vec{r}-\vec{r_0}|-i\omega t}. \tag{13}$$

Equation (13) satisfies the inhomogeneous Helmholtz equation, for which the delta function on the RHS represents a point source of strength q at position $\vec{r_0}$ such that

$$(\nabla^2 + k^2)p = -q\delta(\vec{r} - \vec{r_0}) \tag{14}$$

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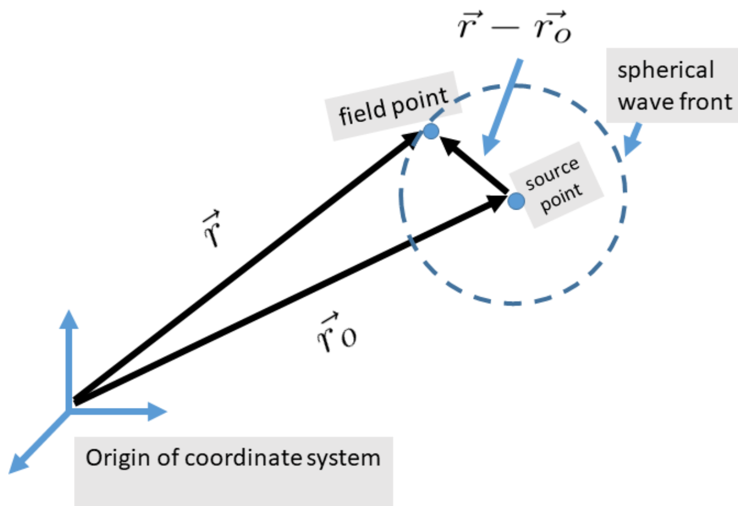

Figure 3: An acoustic source at the source point $\vec{r_0}$ producing the acoustic field at field point $\vec{r}$.

Here are the key properties of the delta function $\delta(\vec{r} - \vec{r_0})$:
(1) $\delta(\vec{r} - \vec{r_0}) = 0$ for $\vec{r} \neq \vec{r_0}$
(2) $\int_V \delta(\vec{r} - \vec{r_0})dV = 1$
(3) $\int_V f(\vec{r})\delta(\vec{r} - \vec{r_0})dV = f(\vec{r_0})$ which is known as the "sifting property" of the delta function. See also Fahy (2001).

We further compress notation by defining $R = |\vec{r} - \vec{r_0}|$, such that

$$g = \frac{e^{ikR}}{4\pi R} \tag{15}$$

and call g the *free space* Green's function (Pierce 1989, Morse and Ingard, 1968) because g satisifies

$$(\nabla^2 + k^2)g = -\delta(\vec{r} - \vec{r_0}) \tag{16}$$

in an *unbounded* medium. By *unbounded* medium we mean there are no nearby boundaries to reflect sound, and therefore the sound spreads uniformly away from the source while decaying in amplitude as $\sim 1/R$, where R is range from source.

A purely unbounded medium might be represented by two people having a conversation–each in their separate helium balloons far above land. But approximately unbounded media are

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everywhere. An excellent one you might experience this winter is being on snow and listening to sounds or speaking with someone nearby– the air above is unbounded and the snow below is highly absorptive of sound, hence sound reflection from the snow boundary is very weak. The opposite effect is experienced by having a conversation inside a stairwell where there are multiple echoes from the nearby reflective walls. Clearly the data from underground garage demo of Lecture 1 was not recorded in an unbounded medium

Note the physical dimension of g is $1/L$. As currently constructed, g embodies all the range-dependent and phase properties of a sound field with point source located at $\vec{r_0}$, but to bring a more useful dimension of pressure, g must be multiplied by some calibration constant.

To summarize, the function g given here represents a sound source (to within a calibration constant) that is concentrated at a point in the manner of a delta function in space, and g is as a solution is known as the *Green's function* for the problem at hand (Frisk, 1994). A formal proof of this solution is given at the end of these notes. This solution can either be a *harmonic* or an *impulsive* Green's function, depending on the time function characteristic of the source, $e^{\pm i\omega t}$ or $\delta(t)$.

A Green's function concentrated in space and impulsive time is discussed in Pierce (1989), see also Tolstoy (1973). In this course we use primarily harmonic Green's function solutions, representing a single-frequency, or narrow band condition, and by Fourier superposition we can combine multiple frequencies to obtain a pulse of time duration τ and bandwidth $\sim 1/\tau$.

Finally, notice that since $|\vec{r} - \vec{r_0}| = |\vec{r_0} - \vec{r}|$ then one can exchange the field point and the source point with the result unchanged. This important property is call *reciprocity*, and the reciprocity principal is often exploited for calibrating microphones and hydrophones (Kinsler *et al.*, 1982). Furthermore, we no longer need to stick with spherically symmetric coordinates. For example, $\vec{r}$ and $\vec{r_0}$ are easily identified in Cartesian coordinates, as in $\vec{r} = [x, y, z]$ and $\vec{r_0} = [x_0, y_0, z_0]$.

We discuss the effect of boundaries, or boundary conditions, in later lectures. For example a major boundary condition to address with a sound source underwater is presence of sea surface and seabed boundary.

Acoustically compact source

Following the exercise concerning the $ka << 1$ we arrive an extraordinarily useful rule: if the characteristic scale L of source is such that $L << \lambda$ where λ is the acoustic wavelength, then the source is *acoustically compact*. Once the source is deemed acoustically compact the scale L is no longer relevant.

The source can be modeling as Eqs. (10) or (13) where the source strength, q is determined empirically by measurement. For example, if p_{rms} is measured at range R m from the source, then we can estimate $|q|$ as follows

$$\frac{|q|}{4\pi}\frac{1}{\sqrt{2}}\frac{1}{R} = p_{rms} \tag{17}$$

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giving at least a value for $|q|$. Often that is all we need anyway, as the real physics relating to sound propagation is embodied in Green's function g.

Lecture 8 Appendix: Formal proof of the Green's function g being a solution to the Helmholtz equation for a point source of sound

Let us next prove that g satisfies Eq.(16). First put the point source location $\vec{r_0}$ at the center of a coordinate system with no loss of generality. Then examine $g = \frac{e^{ikr}}{4\pi r}$ as a solution to

$$\nabla^2 g + k^2 g = -\delta(r) \tag{18}$$

where r is now an ordinary radial coordinate from the origin and there is no need to vectorize.

Now consider a volume V that does not include the origin; under these circumstances we have $\nabla^2 g + k^2 g = 0$ in view of the properties of the delta function. The fact that g, a spherically symmetric wave so defined, is a solution to this *homogeneous* Helmholtz equation is already a settled issue. For example one can put $G = rg$ and G will be a plane wave solution as demonstrated previously.

Next we show that

$$\nabla^2 \frac{e^{ikr}}{r} + k^2 \frac{e^{ikr}}{r} = -4\pi\delta(r) \tag{19}$$

over a small volume V that encloses the source at the origin. Set this up as follows:

$$\int_V \nabla^2 \frac{e^{ikr}}{r} dV + k^2 \int_V \frac{e^{ikr}}{r} dV = -4\pi \tag{20}$$

where the -4π again emerges from the basic property of the delta function.

Examine the two volume integrals separately, put the first equal to I_1 and the second equal to I_2. For I_1 use the divergence theorem to convert the I_1 volume integral into a surface integral giving

$$I_1 = \int_{A_\epsilon} \vec{n} \cdot \nabla \frac{e^{ikr}}{r} dA \tag{21}$$

where A_ϵ is area of a "very small" sphere that encloses the source point. Carefully lay out this surface integral as

$$I_1 = \int_0^{2\pi} d\phi \int_0^{\pi} [\frac{d}{dr} \frac{e^{ikr}}{r}] \epsilon^2 \sin\theta d\theta \tag{22}$$

the factor $[\frac{d}{dr}\frac{e^{ikr}}{r}]$ is evaluated at $r = \epsilon$, and observe that this will be $\frac{ik\epsilon - 1}{\epsilon^2}$. Thus in the limit of $\epsilon \to 0$ we find $I_1 = -4\pi$.

For I_2, recognize that dV equals $d\phi\epsilon^2 \sin\theta d\theta$ and thus this integral will equal 0 as $\epsilon \to 0$. Therefore, Eq. (18) is satisfied.

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